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Periodic Solutions of Some Vector Retarded Functional Differential Equations

J. MAWHIN

Institut Mathématique, Université Catholique de Louvain,
B-1348 Louvain-la-Neuve, Belgium

Submitted by Kenneth Cooke

1. INTRODUCTION

The existence of T-periodic solutions for the T-periodic vector differential equation

$$x' = A(t)\,x - f(t, x) \tag{1.1}$$

with f such that

$$\lim_{|x|\to\infty} \frac{|f(t, x)|}{|x|} = 0 \tag{1.2}$$

uniformly in t, has been proved by various methods when the matrix $A(t)$ is such that the equation

$$x' = A(t)\,x$$

has no nontrivial T-periodic solution (see the books of Krasnosel'skii [6], Reissig, Sansone, and Conti [14], and Roseau [15]). This result has been extended to the case of the vector retarded functional differential equation by Fennell [2].

When the requirement upon the linear part is not satisfied, supplementary conditions upon f are needed to ensure the existence of T-periodic solutions, as follows immediately from the Fredholm alternative [4] for the special case of f independent of x. Such auxiliary conditions appear in papers of Lazer [7] (for a second order scalar differential equation), Ezeilo [1], Sedsiwy [16], Villari [19], and the author [8] (for a third order scalar differential equation), Sedsiwy [17] and Reissig [11, 12] (for the n-th order scalar case) and the same authors [18, 13] for the corresponding vector case. Also, Fennell [2] has extended Lazer's result and method to the retarded case.

Those results are proved by various methods (Brouwer, Schauder, or Leray–Schauder fixed point theorems) which make the arguments rather tedious. On the other hand, all the equations considered in those papers are characterized by the fact that their linear part only admits constant T-periodic solutions and that their nonlinear part verifies (1.2).

In a recent paper [10], the author has given, as an application of more general results, sufficient conditions for the solvability of operator equations

$$Lx = Nx$$

in function spaces, when the linear part L admits only constant solutions and the nonlinear one N is quasibounded, a concept which extends (1.2). The aim of this paper is to apply those results to the problem of the existence of T-periodic solutions for T-periodic vector retarded functional differential equations of the form

$$x^{(k+1)} + A_1 x^{(k)} + \cdots + A_k x' = f(t, x_t, x_t', ..., x_t^{(k)})$$

(the A_i are $(n \times n)$ constant matrices) which contains the equations studied by Ezeilo, Fennell, Lazer, Reissig, Sedsiwy, and Villari quoted above.

Our approach appears to be very natural for this problem as follows both from the extreme simplicity of the proofs and from the substantial generalizations of preceding results that are obtained, in an unified way, by direct application.

2. Some Equations with a Quasibounded Nonlinearity in Function Spaces

For the reader's convenience we shall summarize in this section a few concepts and results from [10] that will be basic for this paper.

If X and Z are normed spaces, with respective norms $\| \cdot \|_X$ and $\| \cdot \|_Z$, a mapping $F: X \to Z$ will be said *quasibounded* (a concept due to Granas [3]) if the number

$$| F | = \inf_{0<\rho<\infty} \left(\sup_{\|x\|_X \geq \rho} \frac{\| F(x) \|_Z}{\| x \|_X} \right)$$

is finite; in this case, $| F |$ is called the *quasinorm* of F.

Let now X be a (not necessarily proper) vector subspace of the (Banach) space $B(S, \mathbb{R}^n)$ of bounded mappings x between some set S and $\mathbb{R}^n$ $(n \geq 1)$, with a norm satisfying the relation

$$\| x \|_X \geq \sup_{s \in S} | x(s) |$$

($| \ |$ is some norm on $\mathbb{R}^n$), with equality at least for constant mappings. The following propositions are proved in [10]:

PROPOSITION 2.1. *Let* $L: \operatorname{dom} L \subset X \to Z$ *and* $N: X \to Z$ *be, respectively, a linear and a quasibounded mapping verifying the following assumptions:*

(1) $\ker L = \{x \in \operatorname{dom} L$: *$x$ is a constant mapping*$\}$, $\operatorname{Im} L$ *is closed and of codimension n and L has a compact right inverse* $K: \operatorname{Im} L \to \operatorname{dom} L$.

(2) *N is continuous, takes bounded sets into bounded sets and*

$$| N | = 0.$$

(3) $Q: Z \to Z$ *being a (continuous) projector such that* $\operatorname{Im} L = \operatorname{Im}(I - Q)$, *there exists one* $R > 0$ *such that*

$$QNx \neq 0$$

for every $x \in \operatorname{dom} L$ *for which*

$$| x(s)| \geqslant R, \qquad \forall s \in S.$$

(4) *The Brouwer degree*

$$d_B[JQN \mid \ker L, b(0, R) \cap \ker L, 0]$$

is not zero, where $J: \operatorname{Im} Q \to \ker L$ *is any isomorphism,* $JQN \mid \ker L$ *is the restriction of* JQN *to* $\ker L$ *and* $b(0, R)$ *is the open ball of center* 0 *and radius* R *in* X.

Then equation

$$Lx = Nx + w \tag{2.1}$$

has at least one solution for every $w \in \operatorname{Im} L$.

PROPOSITION 2.2. *Let L satisfy conditions of Proposition* 2.1 *and let* $N: X \to Z$ *be such that*

$$Nx = \lim_{j \to \infty} N_j x$$

uniformly in X, where every mapping $N_j: X \to Z$ $(j = 1, 2,...)$ *is continuous and satisfy the following condition:*

(2') $(\forall \epsilon > 0)\, (\exists \gamma > 0)\, (\forall j \in \mathbb{N})\, (\forall x \in X): \| N_j x \|_Z \leqslant \epsilon \| x \|_X + \gamma.$

If, morover, every N_j satisfies conditions (3) *and* (4) *of Proposition* 2.1 *with N_j instead of N and R independent of j* $(j = 1, 2,...)$, *then equation* (1.1) *has at least one solution for every* $w \in \operatorname{Im} L$.

3. Some Notations and Preliminary Conventions

If $l \geqslant 0$ is an integer we shall denote by $\mathscr{P}_T{}^l$ the (Banach) space of mappings $x: \mathbb{R} \to \mathbb{R}^n$ which are continuous and T-periodic together with their first l derivatives with the norm

$$\| x \|_l = \sum_{j=0}^{l} [\sup_{t \in \mathbb{R}} | x^{(j)}(t)|]$$

($x^{(j)} = d^j x/dt^j$ and $| \cdot |$ is some norm on $\mathbb{R}^n$).

For some $r \geqslant 0$, let $\mathscr{C}_r$ be the (Banach) space of continuous mappings $\varphi: [-r, 0] \to \mathbb{R}^n$, with the norm

$$\| \varphi \| = \sup_{\theta \in [-r,0]} | \varphi(\theta)| .$$

When $r = 0$, $\mathscr{C}_r$ will be naturally identified with $\mathbb{R}^n$.

Now, if $x \in \mathscr{P}_T{}^l$, $t \in \mathbb{R}$ and $0 \leqslant j \leqslant l$ is an integer, we shall denote as usual [5] by $x_t^{(j)}$ the element of $\mathscr{C}_r$ defined by

$$x_t^{(j)}: [-r, 0] \to \mathbb{R}^n, \qquad \theta \mapsto x^{(j)}(t + \theta).$$

It is to be noted that, for every $0 \leqslant j \leqslant l$,

$$\| x_t^{(j)} \| = \sup_{\theta \in [-r,0]} | x^{(j)}(t + \theta)| \leqslant \sup_{t \in \mathbb{R}} | x^{(j)}(t)| \tag{3.1}$$

When $r = 0$, the mapping $x_t^{(j)}$ will be naturally identified with the element $x(t)$ of $\mathbb{R}^n$. Moreover, a constant mapping in $\mathscr{P}_T{}^l$ or in $\mathscr{C}_r$ will be sometimes identified, without explicit mention, with the element of $\mathbb{R}^n$ given by its constant value.

Lastly we shall introduce the projector

$$P: \mathscr{P}_T{}^l \to \mathscr{P}_T{}^l, \qquad x \mapsto T^{-1} \int_0^T x(t)\, dt.$$

It is immediate that

$$\| Px \|_l = \| Px \|_0 \leqslant \| x \|_0 \leqslant \| x \|_l$$

for every $x \in \mathscr{P}_T{}^l$, and that $\operatorname{Im} P$ is the subspace of $\mathscr{P}_T{}^l$ of constant mappings.

4. Differential Operators with Constant Coefficients in Spaces of T-Periodic Mappings

If $k \geqslant 0$ is an integer, let us now summarize some properties of the vector differential operator with constant coefficients L defined by

$$Lx = x^{(k+1)} + A_1 x^{(k)} + \cdots + A_k x' + A_{k+1} x, \tag{4.1}$$

where the A_i $(i = 1,..., k + 1)$ are $(n \times n)$ constant matrices and

$$\operatorname{dom} L = \{x \in \mathscr{P}_T{}^k \colon x^{(k+1)} \text{ exists and is continuous}\}.$$

It is then clear that

$$\operatorname{Im} L \subset \mathscr{P}_T{}^0.$$

It is well known that the adjoint L^* of L is the operator defined by

$$L^*u = u^{(k+1)} - u^{(k)}(\cdot)\, A_1 + \cdots + (-1)^k u'(\cdot)\, A_k + (-1)^{k+1} u(\cdot)\, A_{k+1},$$

where $u\colon \mathbb{R} \to (\mathbb{R}^n)^*$ is T-periodic and has continuous derivatives up to the order $k + 1$ ($(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$).

The following result is classical [4] and we recall it only for completeness. I_n will denote the $(n \times n)$ identity matrix and $\omega = 2\pi/T$.

PROPOSITION 4.1. *If L is defined by* (4.1), $\ker L \neq \{0\}$ *if and only if the equation*

$$\det(\lambda^{k+1} I_n + \lambda^k A_1 + \cdots + A_{k+1}) = 0 \tag{4.2}$$

has roots of the form $\lambda = im\,\omega$, with m an integer, $\ker L$ *and* $\ker L^*$ *have the same dimension and* $\ker L$ (*respectively* $\ker L^*$) *is formed by the elements of* $\operatorname{dom} L$ (*respectively*, $\operatorname{dom} L^*$) *obtained by taking the real and the imaginary parts of the complex mappings*

$$t \mapsto \exp(im\omega t)\, c \qquad (\textit{respectively},\ t \mapsto \exp(im\omega t)\, d),$$

with $im\omega$ a root of (4.2) *and c the column n-vectors* (*respectively, d the row n-vectors*) *formed by the n first components of the generalized eigenvectors, relative to $im\omega$, of the* $[(k + 1)\, n \times (k + 1)\, n]$ *matrix*

$$A = \begin{pmatrix} 0_n & I_n & 0_n & \cdots & & 0_n \\ \vdots & & & & & \vdots \\ 0_n & 0_n & & \cdots & 0_n & I_n \\ -A_{k+1} & -A_k & & \cdots & -A_2 & -A_1 \end{pmatrix}$$

(*respectively, the n last components of the generalized eigenvectors, relative to $-im\omega$, of minus the transposed of A*). *Lastly,*

$$\operatorname{Im} L = \left\{x \in \mathscr{P}_T{}^0 \colon \int_0^T u(t)\, x(t)\, dt = 0,\ \forall u \in \ker L^*\right\} \tag{4.3}$$

(*Fredholm alternative*) *and there exists a constant $\kappa \geqslant 0$ such that*

$$\| Kx \|_k \leqslant \kappa \| x \|_0 \tag{4.4}$$

for every $x \in \operatorname{Im} L$, *with* K *the* (*unique*) *right inverse of* L *taking values in a fixed topological supplement of* $\ker L$ *in* $\mathscr{P}_T{}^k$.

The following corollary will be particularly useful in the sequel:

COROLLARY 4.1.

$$\ker L = \{x \in \operatorname{dom} L\colon x \textit{ is a constant mapping}\} = \operatorname{Im} P \tag{4.5}$$

if and only if

$$A_{k+1} = O_n \tag{4.6}$$

and equation

$$\det(\lambda^k I_n + \lambda^{k-1} A_1 + \cdots + A_k) = 0 \tag{4.7}$$

has no root λ *of the form* $im\omega$ *with* m *a nonzero integer. In this case,*

$$\operatorname{Im} L = \{x \in \mathscr{P}_T{}^0\colon Px = 0\} \tag{4.8}$$

and the unique right inverse K *of such that* $PK = 0$ *is compact.*

Proof. 1. Necessity. ker L being the subset of dom L of constant mappings, we have

$$A_{k+1} c = 0$$

for every $c \in \mathbb{R}^n$ and hence (4.6) is satisfied. Now, by (4.5) and Proposition 4.1, the equation

$$\det(\lambda^{k+1} I_n + \lambda^k A_1 + \cdots + \lambda A_k) \equiv \lambda^n \det(\lambda^k I_n + \lambda^{k-1} A_1 + \cdots + A_k) = 0 \tag{4.9}$$

has no solution of the form $im\omega$ with m a nonzero integer, and the same is true for Equation (4.7). ker L^* being of dimension n and containing the set of constant mappings from $\mathbb{R}$ into $(\mathbb{R}^n)^*$ coincides with it and then, by taking in (4.3) for u successively the mappings

$$t \mapsto e_i{}^* = (0,\dots, 0, 1, 0,\dots, 0) \qquad (i = 1, 2,\dots, n),$$

we obtain (4.8). To prove the compactness of K, let B be any bounded set in $\operatorname{Im} L$; thus there exists $b \geqslant 0$ such that

$$\| v \|_0 \leqslant b, \qquad \forall v \in B,$$

and hence, by (4.4),

$$\| Kv \|_k \leqslant \kappa b, \qquad \forall v \in B,$$

which shows that the set

$$B' = \{Kv : v \in B\}$$

is bounded in $\mathscr{P}_T^k$. On the other hand, every $Kv \in \operatorname{dom} L$, and thus has continuous derivatives up to the order $k + 1$, and verifies the relation

$$(Kv)^{(k+1)} + A_1(Kv)^{(k)} + \cdots + A_k(Kv)' = v,$$

which implies that

$$\sup_{t \in \mathbb{R}} |(Kv)^{(k+1)}(t)| \leqslant \left[\sum_{j=1}^{k} | A_j | \right] \kappa b + b.$$

Then, by a direct argument using Arzela–Ascoli theorem, B' is relatively compact and hence K is a compact mapping.

2. Sufficiency. By (4.6), $\ker L$ contains the subset of $\operatorname{dom} L$ of constant mappings and, using (4.9) and the condition upon the roots of (4.7), $\ker L$ can only be spanned by constant mappings and is then necessarily equal to (4.5). The remainder of the proof is the same as that of necessity.

5. A Class of Quasibounded Mappings between Spaces of T-Periodic Mappings

We shall introduce in this section a class of quasibounded mappings defined on $\mathscr{P}_T^k$ and related to retarded functional differential equations.

Definition 5.1. We shall say that the mapping

$$f : \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r \to \mathbb{R}^n, \qquad (t, \varphi_1, \ldots, \varphi_{k+1}) \mapsto f(t, \varphi_1, \ldots, \varphi_{k+1}) \tag{5.1}$$

satisfies *condition* (Q) if

$$(\forall \epsilon > 0)\,(\exists \gamma > 0)\,(\forall t \in \mathbb{R})\,(\forall \varphi_i \in \mathscr{C}_r\,, i = 1, \ldots, k+1) : | f(t, \varphi_1, \ldots, \varphi_{k+1})| \leqslant \epsilon(\| \varphi_1 \| + \cdots + \| \varphi_{k+1} \|) + \gamma.$$

Let us now give a few examples of such mappings.

Example 5.1. If there exists $\gamma > 0$ such that

$$| f(t, \varphi_1, \ldots, \varphi_{k+1})| \leqslant \gamma,$$

for every $(t, \varphi_1, \ldots, \varphi_{k+1}) \in \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r$, it is clear that f satisfies condition (Q).

Example 5.2. Let the mapping

$$g: \mathbb{R} \times \mathscr{C}_r \to \mathbb{R}^n, \qquad (t, \varphi) \mapsto g(t, \varphi)$$

be T-periodic with respect to t and such that it takes bounded sets into bounded sets. Then if

$$\frac{| g(t, \varphi)|}{\| \varphi \|} \to 0 \tag{5.2}$$

when $\| \varphi \| \to \infty$, uniformly in t, the mapping f defined by

$$f: \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r \to \mathbb{R}^n, \qquad (t, \varphi_1, \ldots, \varphi_{k-1}) \mapsto g(t, \varphi_1)$$

verifies condition (Q).

The proof is very simple and left to the reader.

Example 5.3. Let

$$h: \mathscr{C}_r \to \mathbb{R}^n, \qquad \varphi \mapsto h(\varphi),$$

be a mapping such that

$$\frac{| h(\varphi)|}{\| \varphi \|} \to 0$$

when $\| \varphi \| \to \infty$. Then, for every $e \in \mathscr{P}_T{}^0$, the mapping f defined by

$$f: \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r \to \mathbb{R}^n, \qquad (t, \varphi_1, \ldots, \varphi_{k+1}) \mapsto e(t) - h(\varphi_1)$$

verifies condition (Q).

It is a special case of Example 5.2.

Now, if the mapping (5.1) is T-periodic with respect to t, we can define the mapping $N: \mathscr{P}_T{}^k \to \mathscr{P}_T{}^0$ by

$$(Nx)(t) = f(t, x_t, x_t', \ldots, x_t^{(k)}) \tag{5.3}$$

and we have the following

PROPOSITION 5.1. *If f is continuous, N is continuous. If f satisfies condition* (Q), *then N is quasibounded with a quasinorm*

$$| N | = 0, \tag{5.4}$$

and takes bounded sets into bounded sets.

Proof. The proof of the continuity of N is very simple. Now, if condition (Q) is satisfied for f, we have,

$$(\forall \epsilon > 0)\,(\exists \gamma > 0)\,(\forall t \in \mathbb{R})\,(\forall x \in \mathscr{P}_T^{\,k}) \colon |f(t, x_t, \ldots, x_t^{(k)})|$$
$$\leqslant \epsilon(\| x_t \| + \cdots + \| x_t^{(k)} \|) + \gamma \leqslant \epsilon \| x \|_k + \gamma,$$

and hence

$$\| Nx \|_0 \leqslant \epsilon \| x \|_k + \gamma \tag{5.5}$$

for every $x \in \mathscr{P}_T^{\,k}$. Thus, N takes bounded sets into bounded sets, and (5.4) holds.

6. A Basic Theorem for the Existence of Periodic Solutions of Some Retarded Functional Differential Equations

With notations of the previous sections, let us consider the T-periodic vector retarded functional differential equation

$$x^{(k+1)} + A_1 x^{(k)} + \cdots + A_k x' = f(t, x_t, x_t', \ldots, x_t^{(k)}) \tag{6.1}$$

and let us prove the following basic

THEOREM 6.1. *Let us suppose satisfied the following conditions:*

(a) *Equation* (4.7) *has no root of the form $im\omega$ with m a nonzero integer.*

(b)
$$f = \lim_{j \to \infty} f^j$$

uniformly in $\mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r$, *where the mappings*

$$f^j \colon \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r \to \mathbb{R}^n, \qquad (t, \varphi_1, \ldots, \varphi_{k+1}) \mapsto f^j(t, \varphi_1, \ldots, \varphi_{k+1})$$

are T-periodic with respect to t, continuous and verify condition (Q) *with γ independent of j ($j = 1, 2, \ldots$).*

(c) *There exists $R > 0$ (independent of j) such that*

$$T^{-1} \int_0^T f^j(t, x_t, \ldots, x_t^{(k)})\, dt \neq 0$$

for every $x \in \mathscr{P}_T^{k+1}$ which verifies

$$| x(t)| \geqslant R, \qquad \forall t \in \mathbb{R}, \qquad (j = 1, 2, \ldots).$$

(d) *For every $j = 1, 2, \ldots$, the Brouwer degree*

$$d_B[F^j, B(0, R), 0]$$

at zero of the mapping

$$F^j: \mathbb{R}^n \to \mathbb{R}^n, \qquad a \mapsto T^{-1} \int_0^T f(t, a, 0,..., 0)\, dt$$

with respect to the open ball $B(0, R)$ *is not zero.*

Then, Equation (6.1) *has at least one T-periodic solution.*

Proof. It consists in showing that conditions of Proposition 2.2 are satisfied with $X = \mathscr{P}_T{}^k$, $Z = \mathscr{P}_T{}^0$, $w = 0$,

$$L: x \mapsto x^{(k+1)} + A_1 x^{(k)} + \cdots + A_k x'.$$

and N^j defined in (5.3) with f^j instead of f, and is a trivial consequence of Corollary 4.1 and Proposition 5.1.

7. An Application to Some Ordinary Vector Differential Equations

The first illustration of the use of Theorem 6.1 will concern the ordinary vector differential equation

$$x^{(k+1)} + A_1 x^{(k)} + \cdots + A_k x' + h(x) = e(t), \tag{7.1}$$

where e: $\mathbb{R} \to \mathbb{R}^n$ and h: $\mathbb{R}^n \to \mathbb{R}^n$ verify the following conditions:

(i) *e is T-periodic, continuous and*

$$T^{-1} \int_0^T e(t)\, dt = 0;$$

(ii) *h is continuous and such that*

$$\lim_{|x| \to \infty} \frac{|h(x)|}{|x|} = 0;$$

(iii) *there exist strictly positive numbers* $r_1 ,..., r_n$, *a permutation* $\{i_1 ,..., i_n\}$ *of* $\{1,..., n\}$ *and an integer* $0 \leqslant m \leqslant n$ *such that*

$$h_{i_l}(x)\, x_{i_l} \geqslant 0, \qquad \text{if} \qquad |x_{i_l}| \geqslant r_l \qquad (l = 1,..., m), \tag{7.2}$$

$$h_{i_l}(x)\, x_{i_l} \leqslant 0, \qquad \text{if} \qquad |x_{i_l}| \geqslant r_l \qquad (l = m + 1,..., n). \tag{7.3}$$

We have the following

THEOREM 7.1. *If condition* (a) *of Theorem* 6.1 *and conditions* (i), (ii), (iii) *above are satisfied, then Equation* (7.1) *has at least one T-periodic solution.*

Proof. It consists in the construction, for (7.1), of a sequence of mappings f^j verifying the conditions of Theorem 6.1. Let $q: \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|q(s)| \leqslant 1, \qquad \forall s \in \mathbb{R}$$

and

$$sq(s) > 0, \qquad \forall s \neq 0. \tag{7.4}$$

If we define the mappings $h^j: \mathbb{R}^n \to \mathbb{R}^n$ by

$$h^j_{i_l}(x) = h_{i_l}(x) + j^{-1}q(x_{i_l}), \qquad l = 1,\dots\ m,$$

$$h^j_{i_l}(x) = h_{i_l}(x) - j^{-1}q(x_{i_l}), \qquad l = m+1,\dots, n,$$

and the mappings g^j by

$$g^j: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad (t, x) \mapsto e(t) - h^j(x), \qquad (j = 1, 2,\dots),$$

it is clear that the sequence (g^j) converges to the mapping

$$(t, x) \mapsto e(t) - h(x),$$

uniformly in $\mathbb{R} \times \mathbb{R}^n$.

On the other hand, it follows from Example 5.3 and from assumptions made upon h that, for every $j = 1, 2,\dots$, the mappings

$$f^j: \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n, \qquad (t, x_1,\dots, x_{k+1}) \mapsto g^j(t, x_1)$$

verify condition (Q) with γ independent of j, which implies that assumption (b) of Theorem 6.1 is satisfied.

Moreover, we have, by (7.2), (7.3), and (7.4), if $|x_{i_l}| \geqslant r_l$,

$$h^j_{i_l}(x)\, x_{i_l} = h_{i_l}(x)\, x_{i_l} + j^{-1}q(x_{i_l})\, x_{i_l} > 0, \qquad l = 1,\dots, m, \tag{7.5}$$

$$h^j_{i_l}(x)\, x_{i_l} = h_{i_l}(x)\, x_{i_l} - j^{-1}q(x_{i_l})\, x_{i_l} < 0, \qquad l = m+1,\dots, n. \tag{7.6}$$

Therefore, if $x \in \mathscr{P}^{k+1}_T$,

$$\begin{aligned} T^{-1}\int_0^T f^j_{i_l}[t, x(t),\dots, x^{(k)}(t)]\, dt &= T^{-1}\int_0^T \{e_{i_l}(t) - h^j_{i_l}[t, x(t)]\}\, dt \\ &= -T^{-1}\int_0^T h^j_{i_l}[t, x(t)]\, dt \neq 0 \end{aligned}$$

when, for every $t \in \mathbb{R}$,

$$|x_{i_l}(t)| \geqslant r_l \qquad (l = 1,..., n; j = 1, 2,...).$$

Then, assumption (c) of Theorem 6.1 is satisfied if we take

$$R = \left(\sum_{i=1}^{n} r_i^2\right)^{1/2}.$$

Lastly, for every $a \in \mathbb{R}^n$,

$$T^{-1} \int_0^T f^j(t, a, 0,..., 0)\, dt = -h^j(a),$$

and, using (7.2), (7.3), we easily obtain, by homotopy invariance of Brouwer degree,

$$d_B[-h^j, B(0, R), 0] = (-1)^m \neq 0,$$

which achieves the proof of the theorem.

Comparison of Theorem 7.1 and the corresponding result of Reissig [13], which extends that of Sedsiwy [18], goes as follows:

(i) Reissig's assumption "Equation (4.7) has only roots with negative real parts" is replaced by the weaker one (a) of Theorem 6.1;

(ii) Reissig's assumption "A_k has no pure imaginary eigenvalue" is avoided;

(iii) Reissig's assumption "either $h_i(x)\, x_i \geqslant 0$ for $|x_i| \geqslant r_i$ $(1 \leqslant i \leqslant n)$ or $f_i(x)\, x_i \leqslant 0$ for $|x_i| \geqslant r_i$ $(1 \leqslant i \leqslant n)$" is replaced by the weaker one (iii);

(iv) Assumptions (i) and (ii) of Theorem 7.1 exactly correspond to the remaining ones of Reissig.

On the other hand, Theorem 7.1 contains as a special case a generalization of a theorem of Lazer for a second order scalar differential equation.

COROLLARY 7.1. *If the matrix A_1 has no eigenvalue of the form $im\omega$ with m a nonzero integer (in particular, if A_1 is symmetric) and if h: $\mathbb{R}^n \to \mathbb{R}^n$ and e: $\mathbb{R} \to \mathbb{R}^n$ satisfy conditions* (i) *to* (iii) *of Theorem* 7.1, *then equation*

$$x'' + A_1 x' + h(x) = e(t)$$

has at least one T-periodic solution.

Lazer's result concerns the scalar case, with for h the stronger assumption

$$h(x)\, x \geqslant 0 \qquad \text{if} \qquad |x| \geqslant R.$$

For another generalization of Lazer's theorem, distinct from Corollary 7.1 but also deduced, by another argument, from ideas related to Proposition 2.1, see [9].

8. The Case of a Scalar Functional Differential Equation

In this section we shall study the scalar T-periodic retarded functional differential equation

$$x^{(k+1)} + a_1 x^{(k)} + \cdots + a_k x' = f(t, x_t, \ldots, x_t^{(k)}) \tag{8.1}$$

where the real constants a_i ($i = 1, \ldots, k$) and the mapping

$$f\colon \mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r \to \mathbb{R}, \qquad (t, \varphi_1, \ldots, \varphi_{k+1}) \mapsto f(t, \varphi_1, \ldots, \varphi_{k+1})$$

(where here $\mathscr{C}_r$ is the space of continuous mappings between $[-r, 0]$ and $\mathbb{R}$) satisfy the following assumptions:

(1) *Equation*

$$\lambda^k + a_1 \lambda^{k-1} + \cdots + a_k = 0$$

has no root of the form $im\omega$, *with* m *a nonzero integer and* $T = 2\pi/\omega$.

(2) f *is* T*-periodic with respect to* t, *continuous and satisfies condition* (Q).

(3) *There exists* $R > 0$ *such that*

$$\left[T^{-1} \int_0^T f(t, x_t, \ldots, x_t^{(k)})\, dt\right] \operatorname{sign} x \geqslant 0 \tag{8.2}$$

for every $x \in \mathscr{P}_T^{k+1}$ *such that, for each* $t \in \mathbb{R}$,

$$|x(t)| \geqslant R,$$

(*or the same condition with the reversed sign of inequality in* (8.2)).

We then have the following:

Theorem 8.1. *If assumptions* (1) *to* (3) *are satisfied, then equation* (8.1) *has at least one* T*-periodic solution.*

Proof. Let us prove, say, the case in which (8.2) is satisfied, the other one being proved by taking $-q$ instead of q below. The sequence of functions (f^j) defined by

$$f^j(t, \varphi_1, \ldots, \varphi_{k+1}) = f(t, \varphi_1, \ldots, \varphi_{k+1}) + j^{-1} r^{-1} \int_{-r}^{0} q[\varphi_1(\theta)]\, d\theta \qquad (j = 1, 2, \ldots)$$

(with $q(x_1)$ instead of $r^{-1} \int_{-r}^{0} q[\varphi_1(\theta)]\, d\theta$ when $r = 0$) converges to f uniformly in $\mathbb{R} \times \mathscr{C}_r \times \cdots \times \mathscr{C}_r$ if we take q like in the proof of Theorem 7.1. On the other hand, for every $x \in \mathscr{P}_T^{k+1}$ such that

$$| x(t)| \geqslant R, \qquad \forall t \in \mathbb{R},$$

we have by (8.2) if we suppose moreover that q is increasing,

$$\begin{aligned}\left[T^{-1} \int_0^T f^j(t, x_t, \dots, x_t^{(k)})\, dt\right] \operatorname{sign} x &\geqslant j^{-1} r^{-1} T^{-1} \int_0^T \int_{-r}^0 q[x(t+\theta)]\, d\theta\, dt \\ &\geqslant j^{-1} r^{-1} T^{-1} \int_0^T \int_{-r}^0 q(R \operatorname{sign} x)\, d\theta\, dt \\ &= j^{-1} q(R \operatorname{sign} x) \operatorname{sign} x > 0, \\ &\qquad (j = 1, 2, \dots). \qquad (8.3)\end{aligned}$$

By taking $x = a$, a constant such that $| a | \geqslant R$, it follows from (8.2) that the Brouwer degree of the mapping

$$a \mapsto T^{-1} \int_0^T f^j(t, a, 0, \dots, 0)\, dt$$

with respect to the origin and the open ball $B(0, R)$ is equal to one $(j = 1, 2, \dots)$. Thus, conditions of Theorem 6.1 are satisfied for (8.1) and the proof is complete.

Remark 8.1. When $k = 2$, condition (1) is equivalent to

(1′) $a_1 \neq 0$ or $a_2 \neq m^2\omega^2$ for every nonzero integer m,

and hence Theorem 8.1 extends a result of Villari [19] in the following ways:

1. Theorem 8.1 covers the case of a retarded functional differential equation instead of an ordinary differential one.

2. (1′) replaces the stronger assumption

$$a_1 a_2 \neq 0 \qquad \text{or} \qquad a_2 < 0.$$

3. Condition (Q) for f replaces the stronger assumption

$$| f(t, u, v, w)| \leqslant \gamma \qquad \text{for every } (t, u, v, w) \in \mathbb{R}^4.$$

Remark 8.2. When $k = 1$, condition (1) of Theorem 8.1 is necessarily satisfied and our result generalizes Fennell's extension [2] to a retarded functional differential equation of Lazer's theorem [7] in the following ways:

1. Assumption (2) replaces the stronger one: "f is fonction only of t and x_t and

$$\|\varphi\|^{-1} |f(t, \varphi)| \to 0 \qquad \text{if} \qquad \|\varphi\| \to \infty.$$

2. Assumption (3) replaces the stronger one: "There exists $R > 0$ such that

$$T^{-1} \int_0^T f(t, x_t)\, dt \geqslant 0 \qquad \text{for every } x \in \mathscr{P}_T{}^2$$

such that

$$x(t) \geqslant R, \qquad \forall t \in \mathbb{R},$$

$$T^{-1} \int_0^T f(t, x_t)\, dt \leqslant 0 \qquad \text{for every } x \in \mathscr{P}_T{}^2$$

such that

$$x(t) \leqslant -R, \qquad \forall t \in \mathbb{R}\text{"}$$

which is equivalent to (3) with the sign "$\geqslant$" in (8.2).

COROLLARY 8.1. *If the following assumptions are satisfied*:

(a) *Condition* (1) *of Theorem* 8.1;

(b) $e\colon \mathbb{R} \to \mathbb{R}^n$ *is continuous*, T*-periodic and*

$$T^{-1} \int_0^T e(t)\, dt = 0;$$

(c) $h\colon \mathscr{C}_r \to \mathbb{R}^n$ *is continuous, takes bounded sets into bounded sets and*

$$\|\varphi\|^{-1} |h(\varphi)| \to 0 \qquad \text{if} \qquad \|\varphi\| \to \infty;$$

(d) *There exists* $R > 0$ *such that either* $h(\varphi)\, \varphi(\theta) \geqslant 0$ *or* $h(\varphi)\, \varphi(\theta) \leqslant 0$ *for every* $\varphi \in \mathscr{C}_r$ *satisfying*

$$|\varphi(\theta)| \geqslant R, \qquad \forall \theta \in [-r, 0],$$

then equation

$$x^{(k+1)} + a_1 x^{(k)} + \cdots + a_k x' + h(x_t) = e(t)$$

has at least one T*-periodic solution.*

Proof. It is a special case of Theorem 8.1 with

$$f(t, x_t, \ldots, x_t^{(k)}) = e(t) - h(x_t).$$

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